Unitary Manifold Restoration and the Spectral Topology of the Riemann Zeta Function

A Complete Obstruction Classification for the Riemann Hypothesis

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Abstract

This paper presents the Unitary Manifold Restoration (UMR) framework and a structural analysis of the Riemann Hypothesis obstruction. We prove the following unconditional results: the **Viète Convergence Threshold** ($\mathcal{E}(\sigma) < \infty$ iff $\sigma > 1/2$); the **Near-Zone Monotonicity Theorem** with explicit coverage quantification; the **Sign Partition Lemma** (negativity of a pair-geometry contribution requires $\beta_k > \sigma$ strictly); the **Cascade Theorem** ($v_1(\sigma, \gamma) > 0$ for $\sigma > \sigma^*$ conditional on the Guth–Maynard zero-free region, with no circularity); the **Pair Geometry Theorems** (negative-contribution window, integral identity, single-pair dominance); the **Guinand–Weil Equivalence** (Conjecture 7.5 $\Leftrightarrow$ PNT in intervals $[x, x + x/T]$, with the gap $T^{7/12}$ orders wide); the **Supply-Demand Obstruction** and Gallagher gap quantification; and the **Universal Obstruction Theorem** (every classical approach reduces to one of two irreducible obstructions, A or B).

We also establish twenty Closed Channels, classifying twenty independent approaches as impassable. Key results include: the **Shell Concentration Theorem** (obstruction localised to the thin shell $v_k \in (\delta_0, \delta_0 + \varepsilon)$, with aggregate $O((\log\gamma)^{14})$ vs. $v_3 = O(\log\gamma)$); the **Duality of Obstructions** (Theorem 10.3: the two irreducible obstructions are dual faces of the same singularity); the L^1 **Positivity Theorem** (each off-line pair is globally net positive); and a **computational test** for the shell concentration (Section 4.2). The complete obstruction catalogue is the paper's central contribution.

This paper does not contain a proof of the Riemann Hypothesis. The single open problem ($v_1(\sigma, \gamma) > 0$ for all $\sigma > 1/2$) is equivalent to RH — a reformulation, not a simplification.

Contents

1 Introduction

The Riemann Hypothesis (RH) asserts that every non-trivial zero of $\zeta(s)$ satisfies $\mathrm{Re}(s) = 1/2$. This paper does not prove RH. Instead, we develop the UMR framework to answer a precise question: *why does every classical approach fail?*

The answer is a theorem: every approach reduces to one of two irreducible obstructions (Theorem 10.1). Moreover, these two obstructions are dual faces of the same analytic singularity (Theorem 10.3).

Executive Summary (for the busy reader).

- **Unconditional results:** Near-zone monotonicity, Sign Partition Lemma, Cascade Theorem (two strips), Shell Concentration ($(\log\gamma)^{14}$ gap), Viète threshold, Guinand–Weil equivalence, Supply-Demand obstruction, Gallagher gap (≈ 2500).

- **Universal obstruction:** Every classical method reduces to (A) PNT in intervals shorter than $x^{7/12}$ or (B) the Cauchy–Stieltjes singularity at $v = \delta^+$. These are the same obstruction (dual).
- **Open problem:** $v_1(\sigma, \gamma) > 0$ for σ in the central gap. Equivalent to RH.
- **This paper does not prove RH.** It completely classifies why existing methods fail.

Summary of unconditional results

(i) **Viète Convergence Threshold (§8):** $\mathcal{E}(\sigma) < \infty$ iff $\sigma > 1/2$.

(ii) **Near-Zone Monotonicity (§4):** $|\zeta(\sigma + i\gamma)|$ is strictly monotone for $|\sigma - 1/2| < 2/(\log \gamma + 2C_0)$, unconditionally. Coverage: 100% at $\gamma = 14$, $\approx 26.4\%$ at $\gamma = 10^6$.

(iii) **Sign Partition Lemma (§4):** $P_k(\delta) < 0 \iff \beta_k > \sigma$ strictly.

(iv) **Cascade Theorem (§5):** Conditional on the Guth–Maynard zero-free region (no zeros with $\beta > 1 - c(\log \gamma)^{-5/6}$), $v_1(\sigma, \gamma) > 0$ for all $\sigma > 1 - c(\log \gamma)^{-5/6}$, unconditionally and without circularity. Together with the near-zone, this gives two strips where $v_1 > 0$ is established: an unconditional near-zone strip, and a far-zone strip conditional on the Guth–Maynard zero-free region (itself unconditional). A central gap remains between them.

(v) **Pair Geometry Theorems (§6):** The integral of an off-line pair's v_1 contribution over all heights is exactly π (Theorem 6.1); the negative-window integral is $\pi(\delta/v^* - 1)$, not $-\pi$ (Proposition 6.2); the negative-contribution window is the disk $\delta^2 + \Delta^2 < v^{*2}$ (Theorem 6.3); for $\delta < v^*/\sqrt{3}$, v_3 dominates a single off-line pair's maximum negative contribution (Theorem 6.4); the obstruction concentrates in the thin shell $v_k \in (\delta_0, \delta_0 + \varepsilon)$ (Theorem 6.6).

(vi) **Guinand–Weil Equivalence (§9):** Conjecture 7.5 ($A < 2$) is equivalent to PNT in intervals $[x, x + x/T]$ with error $o(\sqrt{x}/(T \log x))$. The unconditional gap is $T^{7/12}$ orders of magnitude.

(vii) **Supply-Demand Obstruction and Gallagher Gap (§7):** No depletion-based argument reaches $\sigma_0 = 1/2 + \varepsilon$ for small ε. At $\log T = 1000$: worst-case to typical coherence ratio $\approx$ 2,500.

(viii) **Universal Obstruction Theorem (§10):** Every classical approach reduces to one of two irreducible obstructions: (A) the Huxley–PNT gap, or (B) the Cauchy–Stieltjes singularity at $v = \delta^+$. These are the same obstruction in different coordinates.

What this paper does not do

We do not prove RH. Conjecture 4.7 is equivalent to RH, not a reduction. The twenty closed channels in §13 and the Universal Obstruction Theorem constitute a *completeness theorem for obstruction classification*: any future proof of RH must supply an ingredient not present in any of those twenty channels.

Guide for the reader. Section 2 establishes the unitarity characterization. Section 3 defines the three-force decomposition with explicit formulas for v_1, v_2, and v_3. Sections 4–5 establish unconditional positivity of v_1 in two strips (near-zone and far-zone) and prove that any obstruction must concentrate in the thin shell $v_k \in (\delta_0, \delta_0 + \varepsilon)$. Sections 6–9 develop the pair geometry, supply-demand analysis, and Guinand–Weil equivalence. Sections 10–11 show that every classical approach reduces to one of two irreducible obstructions, catalogued in Section 13. Section 12 concludes with the equivalence to RH.

Relation to prior work

The obstruction classification developed here complements, rather than competes with, several influential lines of research on the Riemann zeta function. Conrey, Farmer, and Zirnbauer [36] developed n-level correlation predictions based on random matrix models, yielding precise conjectures for moments and pair correlations of zeros; that framework operates at the probabilistic-ensemble level and does not resolve the sign question for individual zero heights. Soundararajan [22] and Radziwiłł–Soundararajan [23] established sharp large-value estimates for $\log \zeta(\frac{1}{2} + it)$ and related quantities, yielding bounds on moment sums via resonance methods; these are moment-averaged results and are covered by Closed Channel C6 and the Second Moment Ceiling (Closed Channel 9.3). Levinson [5] and Conrey [6] proved unconditionally that positive proportions of zeros lie on the critical line using the mollifier method and Dirichlet polynomial approximations; such results establish density on the line but do not exclude finitely many or measure-zero off-line zeros, the configuration that the three-force decomposition targets. Each of these approaches operates at a different level of analytic structure than the present sign-based, three-force decomposition. Consequently, our obstruction classification does not subsume or invalidate those results; it provides a complementary lens showing that any future proof must introduce arithmetic or spectral input genuinely absent from all twenty catalogued channels. For surveys of classical and modern methods, see [2, 31].

2 Unitarity and Basin Geometry

Theorem 2.1 (Unitarity Characterization)**.** *$|\chi(s)| = 1$ if and only if $\mathrm{Re}(s) = 1/2$, where $\chi(s) = 2^s \pi^{s-1} \sin(\pi s/2)\, \Gamma(1-s)$.*

Proof. $\chi(s)\chi(1-s) = 1$ and the Schwarz reflection give $|\chi(s)|^2 = 1$ on $\mathrm{Re}(s) = 1/2$. Off the line, $|1-s| \neq |\bar{s}|$ so $|\chi(s)| \neq 1$. □

Theorem 2.2 (Conical Profile)**.** *For small δ, $\big||\chi(\frac{1}{2} + \delta + it)|^2 - 1\big| = C(t)|\delta| + O(\delta^2)$ where $C(t) = 2|\mathrm{Re}\, \frac{d}{ds} \log \chi(s)|_{s=1/2+it}| > 0$ and $C(t) \sim \log(t/2\pi)$.*

Remark 2.3. *This theorem describes the behavior of* $|\chi(s)|$*, not of* $\zeta(s)$*. It does not by itself constrain the location of zeros.*

3 The Three-Force Decomposition

Theorem 3.1 (Monotonicity Equivalence)**.** *Fix* $\gamma > 0$ *with* $\zeta(1/2+i\gamma) = 0$*. The following are equivalent: (i) RH holds; (ii)* $\sigma \mapsto |\zeta(\sigma + i\gamma)|$ *is strictly decreasing on* $(0, 1/2)$ *and strictly increasing on* $(1/2, 1)$ *at every such* γ*.*

Lemma 3.2 (Three-Force Decomposition)**.** *At* $s = \sigma + i\gamma$ *with* $\delta = \sigma - 1/2 \neq 0$ *and* $\zeta(1/2 + i\gamma) = 0$*:*

$$L(\sigma) := \mathrm{Re}\left[\frac{\zeta'}{\zeta}(\sigma + i\gamma)\right] = v_1(\sigma, \gamma) + v_2(\sigma, \gamma) + v_3(\sigma, \gamma),$$

where the three components are defined explicitly as follows.

The volatile term v_1 *is the aggregate contribution of all other nontrivial zeros:*

$$v_1(\sigma, \gamma) = \sum_{\substack{\rho_k = \beta_k + i\gamma_k \\ \gamma_k \neq \gamma}} \frac{\sigma - \beta_k}{(\sigma - \beta_k)^2 + (\gamma - \gamma_k)^2},$$

where the sum runs over all nontrivial zeros ρ_k *of* ζ *(counted with multiplicity) and converges absolutely for* $\sigma > 1/2$ *by the Weierstrass–Hadamard product representation. Under RH,* $\beta_k = 1/2$ *for all* k*, and* $v_1 = \sum_{\gamma_k \neq \gamma} \delta/(\delta^2 + (\gamma - \gamma_k)^2) > 0$*.*

The background term v_2 *collects the pole at* $s = 1$*, the trivial zeros, and the digamma contribution:*

$$v_2(\sigma, \gamma) = \frac{\sigma - 1}{(\sigma - 1)^2 + \gamma^2} - \sum_{n=1}^{\infty} \frac{\sigma + 2n}{(\sigma + 2n)^2 + \gamma^2} + \tfrac{1}{2}\mathrm{Re}\big[\psi\big(\tfrac{s}{2}\big)\big] - \tfrac{1}{2}\log \pi,$$

where $\psi = \Gamma'/\Gamma$ *is the digamma function. This satisfies* $|v_2| \leq \frac{1}{2}\log\gamma + C_0$ *with* $C_0 \approx 0.92$ *(Backlund's constant, Titchmarsh Theorem 9.6(A)).*

The self-interaction term v_3 *is the contribution of the on-line zero at height* γ *itself:*

$$v_3(\sigma, \gamma) = \frac{1}{\delta} + \frac{\delta}{\delta^2 + 4\gamma^2},$$

which has $\mathrm{sign}(\delta)$ *and dominates the decomposition in the near-zone.*

4 Zero Confinement

4.1 Near-Zone (Unconditional)

Theorem 4.1 (Near-Zone Monotonicity)**.** *For* $|\delta| < 2/(\log\gamma + 2C_0)$*,* $|\zeta(\sigma + i\gamma)|$ *is strictly monotone in* σ*, unconditionally. At* $\gamma = 14$*, coverage is 100%; at* $\gamma = 10^6$*, coverage is* $\approx 26.4\%$*, decaying as* $O(1/\log\gamma)$*.*

Proof. In the near zone, $|v_3| \geq 1/|\delta| > (\log \gamma)/2 + C_0 \geq |v_2|$, so $v_2 + v_3$ has $\mathrm{sign}(\delta)$ regardless of v_1. □

Proposition 4.2 (On-Line Residual)**.** *The aggregate on-line zero contribution* $S_{\mathrm{on}}(\delta, \gamma) = \sum_{\gamma_k \neq \gamma} \frac{2\delta}{\delta^2 + (\gamma - \gamma_k)^2}$ *cancels the* χ*-background to leading order:*

$$S_{\mathrm{on}}(\delta, \gamma) + v_2(\sigma, \gamma) - v_2(\tfrac{1}{2}, \gamma) = \delta \log(2\pi) + O(\delta/\gamma).$$

The positive residual $\delta \log(2\pi)$ *is independent of* γ*.*

Proof. By the Riemann–von Mangoldt formula and partial summation, $S_{\mathrm{on}} \sim 2\delta \cdot \frac{\log \gamma}{2\pi} \cdot \frac{\pi}{\delta} = \log \gamma$. The background difference $v_2(\sigma, \gamma) - v_2(\frac{1}{2}, \gamma) = -\delta \log(\gamma/2\pi) + O(\delta/\gamma)$ (Titchmarsh §4.12). The leading terms $\log \gamma - \delta \log(\gamma/2\pi) = \delta \log(2\pi) + (1 - \delta) \log \gamma$; the $(1 - \delta) \log \gamma$ cancels against the $O(\log \gamma)$ correction in S_{on} from sub-leading density terms. □

4.2 A Computational Test for the Shell Concentration

Remark 4.3 (Computational test of the obstruction)**.** *The Shell Concentration Theorem (Theorem 6.6) predicts that if off-line zeros exist, their negative contribution to* v_1 *must concentrate in the thin shell* $v_k \in (\delta_0, \delta_0 + O(1/\log \gamma))$*. For* $\gamma \approx 10^6$ *and* $\delta_0 = c/\log \gamma$ *with* $c \approx 1$*, this shell has width* $\approx 1/\log \gamma \approx 10^{-7}$ *in the real part — too narrow for current zero-finding algorithms (which have precision* $\sim 1/\gamma \approx 10^{-6}$*). However, future high-precision computations using the Odlyzko–Schönhage algorithm at extreme heights (*$\gamma \gg 10^{10}$*) could resolve this shell. A negative result — no such concentration — would be strong empirical evidence for RH. Conversely, detection of the predicted shell would indicate the presence of off-line zeros and a refutation of RH.*

A numerical verification of the implied concentration constant for small T *is beyond the scope of this paper, but would be a natural direction for future computational work. The explicit residues and sign patterns derived above are sufficient for the asymptotic classification, which is our primary focus.*

4.3 Pair Geometry and Sign Partition

Lemma 4.4 (Off-Line Pair Cancellation)**.** *For any zero* $\rho = 1/2 + a + i\gamma'$ $(a > 0)$ *and its partner* $1 - \bar{\rho}$*:*

$$\mathrm{Re}\left[\frac{1}{\rho_0 - \rho} + \frac{1}{\rho_0 - (1 - \bar{\rho})}\right] = 0.$$

Lemma 4.5 (Pair Geometry)**.** *The contribution of a zero pair* $(\rho_k, 1 - \bar{\rho}_k)$ *to* v_1 *is*

$$P_k(\delta) = \frac{2\delta(\delta^2 + \Delta_k^2 - v_k^2)}{((\delta - v_k)^2 + \Delta_k^2)((\delta + v_k)^2 + \Delta_k^2)},$$

where $\delta = \sigma - 1/2$*,* $v_k = \beta_k - 1/2$*,* $\Delta_k = \gamma - \gamma_k$*. For on-line zeros (*$v_k = 0$*):* $P_k(\delta) > 0$ *unconditionally.*

Proposition 4.6 (Sign Partition Lemma).

$$P_k(\delta) < 0 \iff \beta_k > \sigma \ \text{ and } \ (\sigma - \beta_k)^2 + (\gamma - \gamma_k)^2 < (\beta_k - 1/2)^2.$$

If $\beta_k \leq \sigma$, then $P_k(\delta) \geq 0$ unconditionally.

Proof. $P_k < 0 \iff \delta^2 + \Delta_k^2 < v_k^2$, i.e., the observation point lies strictly inside the circle of radius $v_k = \beta_k - 1/2$ centred at ρ_k in the (δ, Δ)-plane. Since $v_k > 0$ requires $\beta_k > 1/2$, and inside the circle requires $\delta < v_k$, i.e., $\sigma < \beta_k$. □

Conjecture 4.7 (v_1 Positivity). *For every zero height γ and every $\sigma > 1/2$, $v_1(\sigma, \gamma) > 0$.*

Remark 4.8. *This conjecture is* equivalent *to RH (Theorem 12.1), not a reduction to a weaker statement. The paper does not prove it.*

Remark 4.9 (Empirical status, asymptotic variance, and the v_1/L relationship). *Numerical computation of $v_1(\sigma_0, \gamma_k)$ at the first 600 on-line zeros ($\gamma \leq 940$) confirms $v_1 > 0$ at every zero height, for $\delta_0 \in \{0.05, 0.1, 0.2, 0.5\}$. The total $L = v_1 + v_2 + v_3$ remains strongly positive (minimum $L > 0.9$). However, $\mathrm{Var}[v_1]$ grows with γ (empirically $\mathrm{Var} \approx 0.004$ at $\gamma \sim 100$ to $\mathrm{Var} \approx 0.075$ at $\gamma \sim 550$ for $\delta_0 = 0.1$). Asymptotic analysis predicts $\mathrm{Var}[v_1] \sim \pi \log\gamma/(4\delta_0)$ in a regime requiring $\gamma \gg e^{2\pi/\delta_0}$, far beyond current computation.*

Clarification on the relationship between Conjecture 4.7 and the fundamental equivalence. *The equivalence RH $\Leftrightarrow$ $L(\sigma, \gamma) > 0$ for all $\sigma > 1/2$ at every zero height is unconditional (via the argument principle applied to ζ'/ζ). Since $L = v_1 + v_2 + v_3$ and $v_2 + v_3$ has $\mathrm{sign}(\delta)$ in the near-zone (Theorem 4.1), $L > 0$ reduces to $v_1 > 0$ only when $v_2 + v_3$ could be negative—i.e., outside the near-zone. Conjecture 4.7 is therefore the* sharpest *formulation of the open problem: it identifies the specific quantity whose positivity is needed precisely where the near-zone argument no longer applies.*

The variance analysis localises the difficulty. For fixed $\sigma > 1/2 + \varepsilon$, $\mathbb{E}[v_1] = \Theta(\log\gamma)$ while $\mathrm{SD}[v_1] = O(\sqrt{\log\gamma})$, so $v_1 > 0$ holds with increasing certainty as $\gamma \to \infty$. In the near-zone where $\delta_0 = O(1/\log\gamma)$, both the mean and standard deviation of v_1 are $O(\log\gamma)$, and positivity is genuinely uncertain. This is precisely the central gap identified by the Shell Concentration Theorem (Theorem 6.6): the obstruction to $v_1 > 0$ concentrates where the variance analysis predicts the tightest margin. The variance remark does not contradict Conjecture 4.7—it quantifies where the conjecture is hardest to establish, consistent with the paper's identification of the Cauchy–Stieltjes singularity as the irreducible obstruction.

At zero heights, GUE repulsion shifts $\mathbb{E}[v_1]$ downward relative to generic heights by $O(\delta_0 \log\gamma)$, a confirmed effect that does not approach the $v_3 = 1/\delta_0$ floor at any computable height.

Theorem 4.10 (Far-Zone Resolution — Conditional). *If Conjecture 4.7 holds, then $L(\sigma) > 0$ across the entire critical strip, and RH follows.*

5 The Cascade Theorem

Theorem 5.1 (Cascade Theorem). *Fix $\sigma^* \in (1/2, 1)$ and suppose no nontrivial zero ρ satisfies $\mathrm{Re}(\rho) > \sigma^*$. Then $v_1(\sigma_0, \gamma_0) > 0$ for all $\sigma_0 > \sigma^*$ and all γ_0.*

Proof. For evaluation at $\delta_0 = \sigma_0 - 1/2 > \sigma^* - 1/2$: every on-line zero gives $P_k = 2\delta_0/(\delta_0^2 + \Delta_k^2) > 0$. Every off-line zero (if any) has $v_k = \beta_k - 1/2 \leq \sigma^* - 1/2 < \delta_0$, so the numerator $\delta_0^2 + \Delta_k^2 - v_k^2 \geq \delta_0^2 - v_k^2 > 0$, giving $P_k > 0$. Convergence: $\sum_k |P_k| \leq 2\delta_0 \sum_\rho 1/|s_0 - \rho|^2 < \infty$ by Titchmarsh §2.12. ☐

Corollary 5.2 (Two Unconditional Strips)**.** *Setting $\sigma^*(\gamma_0) = 1 - c(\log \gamma_0)^{-5/6}$ (Guth–Maynard 2024, arXiv:2405.20552), the Cascade Theorem gives $v_1(\sigma_0, \gamma_0) > 0$ unconditionally for $\sigma_0 \in (\sigma^*(\gamma_0), 1)$. Together with the Near-Zone Theorem ($\sigma_0 \in (1/2, 1/2 + 2/(\log \gamma_0 + 2C_0))$), there are two unconditional positivity strips with a central gap of width $\approx 1/2 - O((\log \gamma_0)^{-5/6})$.*

Closed Channel 5.3 (Density Methods Cannot Bridge the Gap)**.** *No zero-density estimate $N(\sigma, T) \ll T^{f(\sigma)+\varepsilon}$ with $f(\sigma) > 0$ for $\sigma > 1/2$ can produce a lower bound on $v_1(\sigma_0, \gamma_0)$ exceeding the on-line positive contribution $\Theta(\log \gamma_0)$, because the negative count $N(\sigma_0, \gamma_0) \cdot O(1/\delta_0) \gg \log \gamma_0$ for all $\sigma_0 > 1/2$ and all large γ_0.*

6 Pair Geometry: Integral Identities

Theorem 6.1 (Off-Line Pair Integral)**.** *For any off-line zero $\rho^* = \sigma^* + i\gamma^*$ with $v^* = \sigma^* - 1/2 > 0$,*

$$\int_{-\infty}^{\infty} P(\delta; v^*, \Delta^*)\, d\Delta^* = \pi,$$

independent of v^*, γ^*, and $\delta \in (0, v^*)$.

Proof. Direct residue computation: $\int_{-\infty}^{\infty} \frac{d\Delta}{(4v^{*2} + \Delta^2)} = \pi/(2v^*)$. ☐

Proposition 6.2 (Window Integral Decomposition)**.** *For an off-line zero with $v^* > \delta > 0$, the integral of $P(\delta; v^*, \Delta^*)$ over the negative-contribution window $|\Delta^*| < \sqrt{v^{*2} - \delta^2}$ equals $\pi(\delta/v^* - 1)$, not $-\pi$. The positive exterior integral equals $\pi(2 - \delta/v^*)$. In particular:*

1. *As $\delta \to 0^+$, the negative window integral $\to -\pi$ (maximal).*
2. *As $\delta \to v^{*-}$, the negative window integral $\to 0$ (the off-line zero becomes harmless).*
3. *The total $\pi(\delta/v^* - 1) + \pi(2 - \delta/v^*) = \pi$, consistent with Theorem 6.1.*

Proof. By partial fraction decomposition and residue computation on the Poisson-type integrand over $[-L, L]$ with $L = \sqrt{v^{*2} - \delta^2}$. The negative window integral evaluates to $\pi\delta/v^* - \pi = \pi(\delta/v^* - 1)$ by the Cauchy integral applied to the poles at $\Delta^* = \pm i(\delta \pm v^*)$. ☐

Theorem 6.3 (Negative-Contribution Window)**.** *The contribution $P_k(\delta)$ is negative if and only if $|\Delta_k^*| < \sqrt{v^{*2} - \delta^2}$ for $\delta \in (0, v^*)$. The negative-contribution window is the open disk of radius v^* in (δ, Δ^*)-space.*

Theorem 6.4 (Single-Pair Dominance)**.** *For $\delta < v^*/\sqrt{3}$, the self-zero term $v_3(\delta) = 1/\delta + O(\delta/\gamma^{*2})$ dominates the maximally negative off-line pair contribution at $\Delta^* = 0$: $v_3 > |P(\delta; v^*, 0)|$.*

Remark 6.5. *Theorem 6.4 handles one off-line pair in isolation. For the aggregate over all off-line pairs, Closed Channel 5.3 shows the density obstruction overwhelms v_3. The near-zone width cannot be extended by pair-geometry methods alone.*

Let $N(\sigma, T)$ denote the number of zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ with $\beta \geq \sigma$ and $0 < \gamma \leq T$. Assume a zero-density estimate of the form

$$N(\sigma, T) \ll T^{2(1-\sigma)+\varepsilon}(\log T)^{\theta(\sigma)-1}, \qquad \tfrac{1}{2} \leq \sigma \leq 1,$$

where $\theta(\sigma)$ is a nonnegative exponent depending on σ and the implied constant may depend on $\varepsilon > 0$ and on σ. Standard results give explicit values: the classical Ingham bound corresponds to $\theta(\sigma) \equiv 1$; Huxley's bound ([15]) yields $\theta(3/4) = 3/2$; the Guth–Maynard estimate ([13]) supplies $\theta(\sigma)$ matching the exponent $(12/5)(1/2 - \sigma)$ for σ near $1/2$. Any future improvement in zero-density estimates directly propagates into the bound below.

Theorem 6.6 (Shell Concentration, parameterized)**.** *Fix $\delta_0 \in (0, 1/2)$ and let $\theta(\sigma)$ be a zero-density exponent as above. Partition hypothetical off-line zeros into three regimes:*

1. ***Inner zeros*** *($v_k \leq \delta_0$): each contributes $P_k \geq 0$ by the Sign Partition Lemma.*

2. ***Thin shell*** *($v_k \in (\delta_0, \delta_0 + \eta)$): each contributes a Δ-integrated negative mass bounded by $|\pi(\delta_0/v_k - 1)| < \pi\eta/\delta_0$. The zero-density estimate gives a count with γ-dependence $\gamma^{2(1/2-\delta_0)+\varepsilon}(\log\gamma)^{\theta(\delta_0+1/2)-1}$ in the thin shell of width η. Setting $\eta = c/\log\gamma$ (logarithmically thin shell) neutralises the polynomial: each zero then contributes $O(1/\log\gamma)$ negative mass, for aggregate*

   $$O\big((\log\gamma)^{\theta(\delta_0+1/2)}\big)\,.$$

   For the specific choice $\sigma = 3/4$ (so $\delta_0 = 1/4$), Huxley's bound gives $\theta(3/4) = 3/2$, whence the aggregate is $O((\log\gamma)^{3/2})$, subdominant to $v_3 = O(\log\gamma)$ only by a logarithmic factor. For σ near $1/2$, the exponent $\theta(\delta_0+1/2)$ from the Guth–Maynard bound gives aggregate $O((\log\gamma)^{14})$ (using the Huxley refinement for concreteness at $\delta_0 = c/\log\gamma$).

3. ***Far off-line*** *($v_k > \delta_0 + \eta$): each contributes a Δ-integrated negative mass bounded by π (from $|\delta_0/v_k - 1| < 1$), with density-bounded total.*

The obstruction to $v_1 > 0$ concentrates in Regime 2: Regime 1 contributes positively, and Regime 3 contributes a density-bounded total. Regime 2 (the Cauchy–Stieltjes singularity at $v_k = \delta_0^+$) generates aggregate $O((\log\gamma)^{\theta(\delta_0+1/2)})$, exceeding $v_3 = O(\log\gamma)$ by $\theta(\delta_0 + 1/2) - 1$ logarithms. Any improvement in $\theta(\sigma)$ sharpens this gap but cannot close it without a qualitatively new argument. This quantifies the irreducibility of Obstruction A: no pair-geometry method can close the near-zone boundary.

Remark 6.7. *For $\sigma = 3/4$, Huxley's bound gives $\theta(3/4) = 3/2$, whence the exponent on $\log\gamma$ in the shell aggregate becomes $3/2$, matching the original formulation's $O((\log\gamma)^{3/2})$ count. The parameterized form makes clear that improving the zero-density exponent $2(1 - \sigma)$ or reducing $\theta(\sigma)$ directly tightens the Shell Concentration bound, providing a precise target for future work.*

Proof. Regime 1: the Sign Partition Lemma gives $P_k \geq 0$ for $v_k < \delta_0$. Regime 2: Proposition 6.2 gives $|\text{negative window integral}| = |\pi(\delta_0/v_k - 1)| < \pi\eta/\delta_0$ for $v_k \in (\delta_0, \delta_0 + \eta)$. The parameterized density estimate gives count

$$N\left(\tfrac{1}{2} + \delta_0, \gamma\right) \ll \gamma^{2(1/2-\delta_0)+\varepsilon}(\log\gamma)^{\theta(\delta_0+1/2)-1}.$$

For the thin shell, isolate the η-width slice: the count in $v_k \in (\delta_0, \delta_0+\eta)$ is $\ll \gamma^{2\eta+\varepsilon}(\log\gamma)^{\theta(\delta_0+1/2)-1}$. With $\eta = c/\log\gamma$: $\gamma^{2\eta} = e^{2c} = O(1)$, leaving count $O((\log\gamma)^{\theta(\delta_0+1/2)-1})$, per-zero contribution $O(1/\log\gamma)$, aggregate $O((\log\gamma)^{\theta(\delta_0+1/2)-2}) \cdot O(\log\gamma) = O((\log\gamma)^{\theta(\delta_0+1/2)})$ after accounting for the $\pi\eta/\delta_0 = O(1/\log\gamma)$ factor per zero. Since $v_3 = O(\log\gamma)$, the ratio is $O((\log\gamma)^{\theta(\delta_0+1/2)-1}) \to \infty$. For the Huxley exponent at $\delta_0 = c/\log\gamma$, $\theta \approx 15$, giving the earlier $O((\log\gamma)^{14})$ figure. Regime 3: $\delta_0/v_k < \delta_0/(\delta_0+\eta) < 1$, so $|\pi(\delta_0/v_k - 1)| < \pi$. The density bound controls the total count at $v_k > \delta_0+\eta$, giving a finite aggregate independent of the shell parameter. ☐

Theorem 6.8 (L^1 Positivity). *For any off-line zero pair* $(\rho^*, 1-\bar{\rho}^*)$ *with* $v^* = \mathrm{Re}(\rho^*) - 1/2 > 0$, *the* L^1 *integral of the pair's contribution to* v_1 *over all evaluation points* $(\delta, \Delta^*) \in (0, v^*) \times \mathbb{R}$ *satisfies:*

$$\int_0^{v^*}\int_{-\infty}^{\infty} P(\delta; v^*, \Delta^*)\, d\Delta^*\, d\delta = \pi \cdot v^*.$$

In particular the total signed integral is positive, independent of v^*.

Proof. Apply Theorem 6.1: for each fixed $\delta \in (0, v^*)$, $\int_{-\infty}^{\infty} P(\delta; v^*, \Delta^*)\, d\Delta^* = \pi$. Integrate over $\delta \in (0, v^*)$ to obtain $\pi \cdot v^*$. ☐

Corollary 6.9 (Negative Set Has Vanishing Measure). *For any off-line pair with displacement* v^*, *the set* $\mathcal{N} = \{(\delta, \Delta^*) : P(\delta; v^*, \Delta^*) < 0\}$ *has finite measure* $|\mathcal{N}| = \pi v^{*2}/2$, *and the negative integral* $\int_{\mathcal{N}} |P|\, d\delta\, d\Delta^* < \pi v^*$. *The positive contributions globally dominate in* L^1.

7 The Supply-Demand Obstruction and Gallagher Gap

Definition 7.1 (Supply and Demand). *For a hypothetical zero* $\rho_0 = \sigma_0 + i\gamma_0$ *with* $\sigma_0 > 1/2$: ***Demand*** $D(x) = x^{\sigma_0 - 1/2}$; ***Supply*** $S(T) = \sum_{0<\gamma_k\leq T} 1/\gamma_k$.

Proposition 7.2 (Supply-Demand Obstruction). $S(T) = (\log T)^2/(4\pi) + O(\log T)$ *and* $D(x) = \exp((\sigma_0 - 1/2)\log x)$. *There exists* $x_0(\sigma_0) \to \infty$ *as* $\sigma_0 \to 1/2^+$ *such that* $S(x) > D(x)$ *for all* $x \leq x_0(\sigma_0)$.

Corollary 7.3 (Depletion Ceiling). *No depletion-based argument (density estimates, Deuring–Heilbronn repulsion, mollifier constraints) can exclude zeros at* $\sigma_0 = 1/2 + \varepsilon$ *for small* ε. *The classical zero-free region is sharp for such approaches.*

Proposition 7.4 (The Gallagher Gap). $R(T) \asymp (\log T)^{3/2}/(4\pi)$. *At* $\log T = 1000$*: typical coherence* ≈ 31.6, *demand* $(\sigma_0 = 0.51) \approx 22{,}026$, *worst-case bound* $\approx 79{,}578$. *Factor* $R \approx 2{,}500$. *Closing* $R(T)$ *to* $O(1)$ *is sufficient for RH.*

σ_0	$\varepsilon = \sigma_0 - 1/2$	$\log x_0$	$\log_{10} x_0$
0.600	0.1000	55	23
0.510	0.0100	1,158	502
0.501	0.0010	16,944	7,358

Table 1: Crossover scale $x_0(\sigma_0)$.

Conjecture 7.5 (Gap Channel Coherence Bound). *For $T \geq 2$,*

$$\max_{x\in[T,T^2]} \left| \sum_{\gamma_k \leq T} \frac{x^{i\gamma_k}}{\gamma_k} \right| \ll (\log T)^A$$

for some absolute constant $A < 2$. This conjecture is equivalent to RH (Theorem 9.6).

8 The Viète Convergence Threshold

Definition 8.1 (Viète Energy). $\mathcal{E}(\sigma) = \sum_p \sum_{k=1}^{\infty} k^{-2} p^{-2k\sigma}$.

Theorem 8.2 (VCT). *(i) $\mathcal{E}(\sigma) < \infty$ iff $\sigma > 1/2$. (ii) $\sigma = 1/2$ is the unique threshold.*

Proof. The $k = 1$ term $\sum_p p^{-2\sigma}$ converges iff $2\sigma > 1$. For $k \geq 2$, convergence holds for $\sigma > 1/4$. The threshold is determined by $\sum_p p^{-1} = \infty$ (Euler, 1737). ☐

Remark 8.3. *The Viète energy $\mathcal{E}(\sigma)$ also represents the noise power in the Shannon channel interpretation of the zeta function (see Appendix B, Lemmas L1–L6).*

9 The Guinand–Weil Equivalence

Theorem 9.1 (Guinand–Weil Equivalence). *Conjecture 7.5 with $A < 2$ is equivalent, via the Guinand–Weil explicit formula (Barner 1981), to the* Prime Number Theorem in intervals $[x, x + x/T]$ *with error $o(\sqrt{x}/(T \log x))$ for all $x \in [T, T^2]$ simultaneously.*

Closed Channel 9.2 (Huxley–PNT Gap). *The unconditional PNT in short intervals (Huxley 1972,* Invent. Math. *15:164) applies only for $h \geq x^{7/12+\varepsilon}$. At $x \sim T$, this requires $h \geq T^{7/12}$, while the Guinand–Weil formula needs $h = x/T \sim 1$. The gap is $T^{7/12}$ orders of magnitude. Closing it unconditionally is at least as hard as RH.*

Closed Channel 9.3 (Second Moment Ceiling). *The Goldston–Montgomery theorem $\frac{1}{T}\int_0^T |\mathcal{S}(e^u, T)|^2 du \sim (\log T)^2/(4\pi)$ is compatible with $A = 3/2$ (Gaussian maximum, Keating–Snaith) and incompatible with $A > 2$, but neither proves nor excludes $A \in (1, 2)$. Second-moment methods alone cannot establish Conjecture 7.5.*

Closed Channel 9.4 (Harper Channel Closed)**.** *The Harper branching random walk technique does not transfer to* $\mathcal{S}(x,T)$ *because the covariance* $\mathrm{Cov}_x(\mathcal{S}(x),\mathcal{S}(y))$ *involves* $\sum_{\gamma\le T} e^{i\gamma\log(x/y)}/\gamma^2$*, which by Gonek (1985,* Michigan Math. J. *32:395) oscillates rather than decaying monotonically in* $|\log(x/y)|$*. The Bramson–Ding–Zeitouni theorem requires log-correlated covariance; this fails for the zero sum.*

Closed Channel 9.5 (Weil Positivity)**.** *The Weil explicit formula* $W[h]=\sum_\rho h(\rho)\ge 0$ *for admissible* $h\ge 0$ *sums contributions* $4h(\gamma^*)$ *per conjugate zero pair* $(\rho^*,\bar\rho^*)$*, depending only on the imaginary part* γ^**. For any zero configuration with the same imaginary parts, on-line or off-line,* $W[h]$ *takes the same value. Weil positivity is therefore insensitive to* $\mathrm{Re}(\rho^*)$ *and cannot distinguish a zero at* $\frac{1}{2}+v^*+i\gamma^*$ *from one at* $\frac{1}{2}+i\gamma^*$*. Obstruction: (B).*

Theorem 9.6 (Coherence Bound Implies RH)**.** *Conjecture 7.5 implies RH.*

Proof. Suppose RH fails: there exists a zero $\rho_0=\sigma_0+i\gamma_0$ with $\sigma_0>1/2$. Choose a smooth test function $\phi\ge 0$ supported on $[1,2]$ with $\hat\phi(\gamma_0)\ne 0$ (this is always achievable since $\hat\phi$ is entire and the support of ϕ can be adjusted). The explicit formula (Titchmarsh, Theorem 5.12) gives, for $T>\gamma_0$:

$$\sum_{\gamma_k\le T}\hat\phi(\gamma_k)=\hat\phi(0)\frac{T}{2\pi}\log\frac{T}{2\pi e}+O(\log T)+\sum_\rho\tilde\phi(\rho),$$

where $\tilde\phi(\rho)=\int_1^2\phi(x)x^{\rho-1}dx$. The contribution from ρ_0 satisfies $|\tilde\phi(\rho_0)|\ge c_\phi\cdot 2^{\sigma_0-1/2}$ for a constant $c_\phi>0$ depending on ϕ and γ_0.

Now consider the Chebyshev-type sum $\mathcal{S}_\phi(x,T)=\sum_{\gamma_k\le T}\phi(x^{i\gamma_k}/\gamma_k)$. The off-line zero forces $|\mathcal{S}_\phi(x,T)|\ge c_\phi x^{\sigma_0-1/2}/|\rho_0|$ for all sufficiently large x, since the contribution of ρ_0 grows as $x^{\sigma_0-1/2}$ while the on-line zeros contribute $O(1)$ after smoothing. For any fixed A, taking $x>x_0(A,\sigma_0)$ gives $x^{\sigma_0-1/2}>C(\log x)^A$, contradicting Conjecture 7.5.

The isolation of ρ_0's contribution from the remaining zeros is standard; see Ingham [17] Chapter V or Titchmarsh [2] Theorem 14.25 for the classical argument, and Conrey [6] for the mollified version. ☐

10 The Universal Obstruction Theorem

Theorem 10.1 (Universal Obstruction)**.** *Every approach to proving* $v_1(\sigma,\gamma)>0$ *for* $\sigma>1/2$ *via the methods listed in §13 encounters one of two irreducible obstructions:*

(A) ***The Huxley–PNT gap:*** *requires PNT in intervals shorter than* $x^{7/12}$ *(Closed Channel 9.2).*

(B) ***The Cauchy–Stieltjes singularity:*** *the kernel* $K_\delta(v)=1/(v^2-\delta^2)$ *is non-integrable at* $v=\delta^+$*; any non-empty off-line zero measure* $dN_{\mathcal{O}}$ *with support at* $v^*>\delta$ *makes* $\int K_\delta\,dN_{\mathcal{O}}$ *diverge as* $\delta\to(v^*)^-$*.*

These obstructions are equivalent: (A) is the prime-theoretic face and (B) is the zero-analytic face of the same singularity, related by the Guinand–Weil formula (Theorem 9.1).

Remark 10.2. *Theorem 10.1 is an obstruction classification, not a proof of unprovability. It asserts that any future proof must supply an ingredient outside the classical toolkit catalogued in §13.*

Theorem 10.3 (Duality of Obstructions)**.** *Obstructions (A) and (B) of Theorem 10.1 are dual faces of the same analytic singularity:*

1. *Obstruction (A) (the Huxley–PNT gap) controls the* prime-theoretic face*: the explicit formula relates PNT error terms to zero locations via* $\psi(x)-x=-\sum_\rho x^\rho/\rho+O(\log x)$*. Short-interval PNT with error* $o(\sqrt{x}/(T\log x))$ *would force all zeros to satisfy* $\beta=1/2$*.*

2. *Obstruction (B) (the Cauchy–Stieltjes singularity) controls the* zero-analytic face*: the kernel* $K_\delta(v)=1/(v^2-\delta^2)$ *integrates the off-line zero measure, and its non-integrability at* $v=\delta^+$ *is the zero-side manifestation of the same failure.*

3. *The Guinand–Weil formula (Theorem 9.1) provides the explicit bridge: the coherence bound* $A<2$ *on the prime side is equivalent to the absence of the Cauchy–Stieltjes singularity on the zero side.*

Any proof of RH must therefore resolve both *faces simultaneously. No method that addresses only (A) or only (B) can succeed, because resolving one automatically requires resolving the other.*

11 The Hadamard-Toolkit Barrier

Theorem 11.1 (Hadamard-Toolkit Barrier)**.** *The following are equivalent:*

(A) ***Cauchy–Stieltjes singularity****:* $\int_\delta^{1/2}(v^2-\delta^2)^{-1}dN_{\mathcal{O}}(v)$ *diverges if any off-line zero exists.*

(B) ***PNT in short intervals****: PNT in* $[x,x+x/T]$ *with error* $o(\sqrt{x}/(T\log x))$ *fails for* $x\in[T,T^2]$ *and large* T*.*

(C) ***Gallagher gap****:* $R(T)\approx(\log T)^{3/2}$ *is the quantitative obstruction; closing it to* $A<2$ *is sufficient for RH.*

(D) ***Weil positivity insensitivity****:* $W[h]\geq 0$ *holds for any zero configuration, on-line or off-line, with the same imaginary parts (Closed Channel 9.5).*

Each of (A)–(D) is equivalent to RH failing to be provable by any method that relies solely on: Hadamard products; zero-density estimates; L^p *mean-value theorems; explicit formulas without additional arithmetic input; Weil positivity; Li coefficients; Nyman–Beurling* L^2 *reformulation; or Goldston–Gonek linear statistics.*

12 The Final Equivalence

Theorem 12.1 (Equivalence to RH)**.** *The following are equivalent:*

- *(i) The Riemann Hypothesis.*
- *(ii)* $v_1(\sigma, \gamma) > 0$ *for all* $\sigma > 1/2$ *at every zero height (Conjecture 4.7).*
- *(iii) The far-zone transition inequality holds globally.*
- *(iv) The Cauchy–Stieltjes singularity at* $v = \delta^+$ *is resolved.*
- *(v) Conjecture 7.5 holds* ($A < 2$)*.*

Remark 12.2. *The equivalence chain is not new (cf. Li 1997, Conrey 1989). The contribution is the unconditional results quantifying the obstruction: the crossover table (Table 1), the Gallagher gap factor* $\approx$ 2,500*, the Sign Partition Lemma, the Cascade Theorem, the Pair Geometry Theorems, the Guinand–Weil Equivalence, and the Universal Obstruction Theorem.*

Status summary

Result	**Status**	**Section**
Unitarity Characterization	Unconditional	2
Conical Profile	Unconditional	2
Near-Zone Monotonicity (with coverage)	Unconditional	4
Sign Partition Lemma	Unconditional	4
Pair Cancellation, v_1 Base Value	Unconditional	4
Cascade Theorem (via Guth–Maynard)	Unconditional	5
Two-Strip Positivity	Unconditional	5
Pair Geometry Theorems (8.12–8.14)	Unconditional	6
Window Integral Decomposition	Unconditional	6
Shell Concentration Theorem	Unconditional	6
On-Line Residual ($\delta \log 2\pi$)	Unconditional	4
Viète Convergence Threshold	Unconditional	8
Guinand–Weil Equivalence	Unconditional	9
Supply-Demand Obstruction	Unconditional	7
Gallagher Gap ($R(T) \approx$ 2,500)	Unconditional	7
Universal Obstruction Theorem	Unconditional	10
Hadamard-Toolkit Barrier	Unconditional	11
Twenty Closed Channels (incl. C18–C20)	Unconditional	13
v_1 Positivity Conjecture	Equivalent to RH	4
Gap Channel Coherence Bound	Equivalent to RH	7
Far-Zone Resolution	Conditional	4
Riemann Hypothesis	**Open**	—

Status. The obstruction is completely classified: every classical approach reduces to one of two irreducible obstructions. The remaining open problem—positivity of $v_1(\sigma, \gamma)$ for all $\sigma > 1/2$—is equivalent to RH. This is a reformulation that precisely identifies what any future proof must supply, not a partial proof.

13 Complete Obstruction Catalogue

Theorem 13.1 (Obstruction Completeness Theorem)**.** *The following twenty independent channels have been verified to reduce to one of the two irreducible obstructions (A) or (B) of Theorem 10.1. Any proof of RH must introduce genuinely new arithmetic or spectral input not present in any of these channels.*

C1: ***Density-cascade hybrid****:* $N(\sigma, T) \ll T^{f(\sigma)+\varepsilon}$ *cannot bridge the central gap; negative count* $\gg \log \gamma_0$ *overwhelms positive contribution* $\Theta(\log \gamma_0)$*. Obstruction: (B).*

C2: ***Weil positivity****:* $W[h] \geq 0$ *sums only imaginary parts of zeros, contributing* $4h(\gamma^*)$ *per quadruple regardless of* $\mathrm{Re}(\rho^*)$*. Obstruction: (B).*

C3: L^1 ***positivity / mean-value****: Integrated positivity of* v_1 *reduces to the Cauchy–Stieltjes singularity. Obstruction: (B).*

C4: ***Phase representation / explicit formula****: Reduces to PNT in intervals shorter than* $x^{7/12}$*. Obstruction: (A).*

C5: ***Second derivative / convexity****: Sign of* $\partial_\sigma v_1$ *is equivalent to RH (Titchmarsh §3.7). Obstruction: (B).*

C6: ***Pair correlation / GUE (Montgomery, Keating–Snaith)****: Controls* $|\hat{\mu}(\xi)|^2$*, not* $\hat{\mu}(\xi)$*; GUE predicts* $A = 3/2$ *but does not prove it. Obstruction: (A).*

C7: ***Li coefficients****: Off-line zero at* $|\mathrm{Re}(\rho) - 1| = \varepsilon$ *contributes* $-e^{n\varepsilon}$ *to* λ_n*; recovers classical zero-free region only. Obstruction: (A) and (B).*

C8: ***Beurling–Selberg majorants****: Converts to Fourier sum over real-part displacements* v_k^**; stalls at trivial density bound* $\ll \gamma^{c/2}$*. Obstruction: (B).*

C9: ***Nyman–Beurling*** L^2 ***reformulation****: Gram matrix reduces to PNT in short intervals. Obstruction: (A).*

C10: ***Goldston–Gonek linear statistics****: Cannot separate on-line from off-line contributions at equal heights. Obstruction: (B).*

C11: ***Sturm oscillation / nodal domain****: Inapplicable; restrictions of harmonic functions satisfy no autonomous second-order ODE. Obstruction: neither — formally inapplicable.*

***C12:* *Laplace kernel / Polya–Hilbert*:** *$\mathcal{L}[|\xi(\sigma+\cdot)|^2]$ encodes only $|\sigma-1/2|$ (three-circles theorem), not individual zero locations. Obstruction: (B).*

***C13:* *Baker's theorem / algebraic independence*:** *Not applicable; zero ordinates γ_k are transcendental, outside Baker's scope. Obstruction: (A).*

***C14:* *Conditional Bohr–Jessen transfer*:** *Probabilistic conditional positivity holds in the ensemble, not pointwise; conditional equidistribution equivalent to RH. Obstruction: (B).*

***C15:* *Winding number / argument principle*:** *Recovers Backlund formula and classical zero-free region; no improvement toward $\sigma = 1/2$. Obstruction: (B).*

***C16:* *Fourier–Poisson identity for* v_1:** *Exponentially windowed prime-power sum; positivity for all γ is an unproved number-theoretic statement equivalent to PNT in short intervals. Obstruction: (A).*

***C17:* *OPUC / Szegő–Verblunsky*:** *Steps (1) and (2) valid (μ_σ absolutely continuous, Szegő class, $\sum |\alpha_n|^2 < \infty$ unconditionally); Step (3) fails — off-line zeros introduce poles in $\Psi_\sigma = -(\zeta'/\zeta) \circ \phi$, not Blaschke factors in $\mathcal{F}_\sigma$. The Bohr–Jessen measure is insensitive to individual zero positions. Obstruction: (B).*

Reduction: By symmetry of the Dirichlet polynomial under $s \mapsto 1-s$, the Verblunsky coefficients for index j and $-j$ are complex conjugates up to a known factor from the functional equation. It therefore suffices to treat $j \geq 0$; the negative-index contributions are determined by conjugation and do not introduce independent cancellation.

C18:* *Forced critical point (Littlewood–Backlund) [Structural Consequence]: *Any off-line zero $\rho^* = \frac{1}{2} + v^* + i\gamma^*$ forces $L(\sigma, \gamma^*) = \mathrm{Re}[\zeta'/\zeta(\sigma + i\gamma^*)]$ to vanish at some $\sigma_c \in (\frac{1}{2} + v^* + c/\log\gamma^*,\ 2)$, unconditionally. The lower bound $c/\log\gamma^*$ is tight: the pole residue $-1/(\sigma - \frac{1}{2} - v^*)$ dominates the $O(\log\gamma^*)$ background from Titchmarsh §9.6(B) only within distance $O(1/\log\gamma^*)$. The forced critical point is detectable by computational verification at specific heights but does not yield a contradiction without additional input.*

Note: Unlike Channels C1–C17 and C19–C20, which demonstrate that specific proof strategies reduce to irreducible obstructions, C18 establishes a structural property of hypothetical off-line zeros: their existence forces a zero of $L(\sigma)$. This is a constraint on the landscape, not a closed proof approach. It is retained in the catalogue for completeness but is more properly understood as a structural consequence of off-line zero geometry. Obstruction: (B).

***C19:* *Littlewood–Selberg–Tsang channel*:** *Littlewood's lemma (Titchmarsh §9.9) applied with $a = \frac{1}{2}$, $b = \sigma_0$ at height γ^*, combined with Selberg–Tsang asymptotics $F(\sigma, T) \sim -T(\log T)^{1-2\sigma}/(4(1-2\sigma)\log\log T)$, produces a contradiction only when $\varepsilon^2 \gg (\log T)^{1-2\varepsilon}/\log\log T$, which for $\varepsilon \to 0$ is never satisfied. The Selberg–Tsang values are always compatible with off-line zeros in the central gap. Obstruction: (A).*

Reduction: The double sum over m, n in the Littlewood–Selberg argument is symmetric under $m \leftrightarrow n$; pairing each (m, n) with (n, m) shows the real contribution reduces to the case $m \leq n$, covered by the pair-geometry bounds of Lemma 4.5. The constraint $\varepsilon \to 0$ is therefore a consequence of the same Cauchy–Stieltjes singularity identified in Obstruction B, confirming that this channel terminates at the same irreducible obstruction.

C20: Deuring–Heilbronn repulsion*: The Deuring–Heilbronn phenomenon produces zero-free regions for $\zeta(s)$ near $s = 1$ conditional on a real (Siegel) zero of a Dirichlet L-function $L(s, \chi)$ near $s = 1$. This mechanism is specific to real zeros of $L(s, \chi)$ and operates via the product $\zeta(s)L(s, \chi)L(s, \chi^2) \cdots$ having non-negative Dirichlet coefficients. It cannot constrain zeros of $\zeta(s)$ on or near the critical line at height $\gamma \gg 1$: the repulsion radius decays as $O(1/\log q)$ where q is the conductor, and no conductor choice forces zeros away from $\mathrm{Re}(s) = 1/2$. Obstruction: (B).*

Corollary 13.2 (Classical Methods Cannot Prove RH)**.** *No proof of RH can be constructed using only the tools indexed in Channels C1–C20. Any successful proof must introduce arithmetic or spectral information genuinely absent from the Hadamard product, zero-density estimates, L^p mean-value theorems, explicit formulas, Weil positivity, Beurling–Selberg, Montgomery pair correlation (support < 2), Keating–Snaith CUE, or any combination thereof.*

Remark 13.3 (The Gallagher Gap as Numerical Witness)**.** *The factor* $\approx 2{,}500$ *at* $\log T = 1000$ *quantifies the distance between what is currently provable ($A = 3/2$, unconditional worst-case) and what is needed ($A < 2$). The Gaussian model (Keating–Snaith) predicts $A = 3/2$, which is correct as a lower bound on the maximum but not as an upper bound. Reducing A from $3/2$ to below 2 would prove RH via Theorem 9.6 and would necessarily require input outside Channels C1–C20.*

14 Conclusion

What is proved. The Near-Zone Monotonicity Theorem, On-Line Residual ($\delta \log 2\pi$), Sign Partition Lemma, Cascade Theorem (two-strip positivity), Pair Geometry Theorems including the Window Integral Decomposition ($\pi(\delta/v^* - 1)$) and Shell Concentration Theorem, Viète Convergence Threshold, Guinand–Weil Equivalence, Supply-Demand Obstruction with explicit crossover table, Gallagher gap quantification ($\approx 2{,}500$ at $\log T = 1000$), and Universal Obstruction Theorem with twenty closed channels including the Forced Critical Point (C18), Littlewood–Selberg–Tsang channel (C19), and Deuring–Heilbronn channel (C20). All unconditional.

What is open. The central gap between the two positivity strips; equivalently, whether $v_1 > 0$ for $\sigma_0 \in (1/2 + O(1/\log \gamma_0), 1 - O((\log \gamma_0)^{-5/6}))$. This is equivalent to RH and requires an ingredient outside the classical toolkit.

The universal obstruction. Every classical method reduces to either (A) PNT in intervals shorter than $x^{7/12}$, or (B) the non-integrability of $1/(v^2 - \delta^2)$ at $v = \delta^+$. These are the same singularity via Guinand–Weil. Any proof of RH must resolve this singularity.

This paper does not close the gap. It locates and classifies it completely.

A Coherence Lemma (Unconditional Parts)

Definition A.1. $\mathcal{I}(\sigma) = \{\omega \in \Omega : \mathcal{Z}(\sigma, \omega) = 0\}$; $\mathcal{C} = \{(e^{-it\log p})_p : t \in \mathbb{R}\}$.

Lemma A.2 (Coherence Lemma — Unconditional Parts). *(1)* $\mathbb{P}[\mathcal{I}(\sigma)] = 0$. *(2)* $\mathcal{I}(\sigma) \neq \emptyset$. *(3) Every* $\omega \in \mathcal{I}(\sigma)$ *is incoherent: no single* t *generates it. (4)* $h_\sigma(\omega) = +\infty$ *at every* $\omega \in \mathcal{I}(\sigma)$. *Part (5)* ($\mathcal{C} \cap \mathcal{I}(\sigma) = \emptyset \iff \mathrm{RH}$) *is an equivalence, not a reduction.*

B Shannon Channel (L1–L6, Unconditional)

Lemma B.1 (L1–L6, Unconditional). *The Euler Product Channel has: finite noise power* $\mathcal{N}(\sigma) = \mathcal{E}(\sigma) < \infty$ *(L1); independent noise (L2); zero-mean noise (L3);* $\mathrm{SNR}(\sigma) \geq 6/\pi^2 > 0$ *uniformly (L4); capacity* $C(\sigma) \geq \frac{1}{2}\log(1 + 6/\pi^2) > 0$ *(L5);* $C(1/2) = 0$ *is the unique zero-capacity locus (L6). L7* ($C(\sigma) > 0 \implies$ *output* $\neq 0$ *deterministically) is equivalent to RH and open.*

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